Math 1510 Week 8

Partial fractions (Useful for integration)

Goal: Express a proper rational function as a sum of simpler ones

Rmk: $\frac{p(x)}{q(x)}$ is proper if $\deg p < \deg q$

eg
$$\frac{x}{x^2+3x+2} = \frac{-1}{x+1} + \frac{2}{x+2}$$

$$\frac{x^2+20x+11}{(x+1)^2(x-3)} = \frac{-4}{x+1} + \frac{2}{(x+1)^2} + \frac{5}{x-3}$$

$$\frac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \underbrace{\frac{-x+7}{x^2+x+1} + \frac{5}{x-2}}$$

Rmk RHS is easier for integration

Recall

ax^2+bx+c is irreducible (cannot be further factorized)

$\iff \Delta = b^2-4ac < 0$

- $x^2-1 = (x+1)(x-1)$ is reducible $(\Delta = 4)$
- x^2+x+10 is irreducible $(\Delta = -39)$

Procedure: Given proper $\frac{p(x)}{q(x)}$

1. Factorize $q(x)$ into a product of linear and irreducible quadratic factors
2. Write down general terms

Factor of $q(x)$	Terms in partial fractions
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c (irreducible)	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

3. Determine the unknown coefficients A_i, B_i in step 2 by substitution or comparing coefficients.

eg $\dfrac{9x-13}{x^2+x-12}$

① $x^2+x-12 = (x+4)(x-3)$

② General terms:

Let $\dfrac{9x-13}{x^2+x-12} = \dfrac{A}{x+4} + \dfrac{B}{x-3}$

③ $\Rightarrow 9x-13 = A(x-3)+B(x+4)$

$= (A+B)x + (-3A+4B)$

Comparing coefficients

$$\Rightarrow \begin{cases} A+B = 9 & \cdots ① \\ -3A+4B = -13 & \cdots ② \end{cases}$$

$3\times① + ② \Rightarrow 7B = 14 \Rightarrow B = 2$

Put $B=2$ into ① $\Rightarrow A=7$

$\therefore \dfrac{9x-13}{x^2+x-12} = \dfrac{7}{x+4} + \dfrac{2}{x-3}$

eg $\dfrac{x^2+20x+11}{(x+1)^2(x-3)}$ ← already factorized

Sol General term:

Let $\dfrac{x^2+20x+11}{(x+1)^2(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-3}$

$x^2+20x+11 = A(x+1)(x-3) + B(x-3) + C(x+1)^2$

We determine coefficients by substitution this time:

Put $x=3$ $\quad 80 = 16C \Rightarrow C=5$

Put $x=-1$ $\quad -8 = -4B \Rightarrow B=2$

Put $x=0$ $\quad 11 = -3A-3B+C = -3A-1$

$\Rightarrow A=-4$

$\dfrac{x^2+20x+11}{(x+1)^2(x-3)} = -\dfrac{4}{x+1} + \dfrac{2}{(x+1)^2} + \dfrac{5}{x-3}$

eg $\dfrac{4x^2+14x-9}{(x^2+x+1)(x-2)}$

$\underbrace{x^2+x+1}$: $\Delta=-3 \Rightarrow$ irreducible

Sol General terms.

$$\frac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}$$

$$4x^2+14x-9 = (Ax+B)(x-2) + C(x^2+x+1)$$

Substitutions (eg $x=2,0,1$) give

3 equations $\xrightarrow{\text{Solve}}$ $A=-1, B=7, C=5$

$$\frac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \frac{-x+7}{x^2+x+1} + \frac{5}{x-2}$$

eg $\dfrac{x^4+3x^2-x+1}{x^5+2x^3+x}$

Sol $x^5+2x^3+x = x(x^4+2x^2+1) = x(x^2+1)^2$

Let $\dfrac{x^4+3x^2-x+1}{x^5+2x^3+x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

By substitution/comparing coefficients + Solving eqn:

$$\frac{x^4+3x^2-x+1}{x^5+2x^3+x} = \frac{1}{x} + \frac{x-1}{(x^2+1)^2} \quad (B=C=0)$$

Rmk If $\dfrac{p(x)}{g(x)}$ is improper ($\deg p \geq \deg g$)

we should do long division first

eg $\dfrac{2x^3}{x^2-1} = 2x + \dfrac{2x}{x^2-1} = 2x + \dfrac{1}{x-1} + \dfrac{1}{x+1}$

Long division

Partial fractions

$\Rightarrow 2x^3 = (x^2-1)(2x) + 2x$

Indefinite Integral

> **Defn** For $f(x)$ defined on an interval,
> $$\int f(x)\,dx = \text{Indefinite integral of } f(x)$$
> $$= \text{Anti-derivative of } f(x)$$

$$g(x) = \int \underbrace{f(x)}_{\text{Integrand}}\,dx \quad \Rightarrow \quad g'(x) = f(x)$$

Integration sign   Integrand   x is integration variable

f is <u>the</u> derivative of g, g is <u>an</u> anti-derivative of f

eg $(x^2)' = 2x$, $(x^2+3)' = 2x$

$\therefore$ Both x^2 and x^2+3 are anti-derivatives of $2x$

Indeed, $(x^2+C)' = 2x$ for any constant C.

$\therefore \int 2x\,dx = x^2 + C$, where C is a constant

> $\int f(x)dx$ is only defined up to an additive constant.
> (integration constant.)

eg Verify that $\int \frac{1}{x}dx = \ln|x| + C$

Sol Need to show $(\ln|x| + c)' = \frac{1}{x}$

For $x \in (0, \infty)$,

$$(\ln|x| + c)' = (\ln x + C)' = \frac{1}{x}$$

For $x \in (-\infty, 0)$,

$$(\ln|x| + c)' = (\ln(-x) + C)' = \frac{1}{-x}(-1) = \frac{1}{x}$$

$$\therefore \int \frac{1}{x}dx = \ln|x| + C$$

Compare :

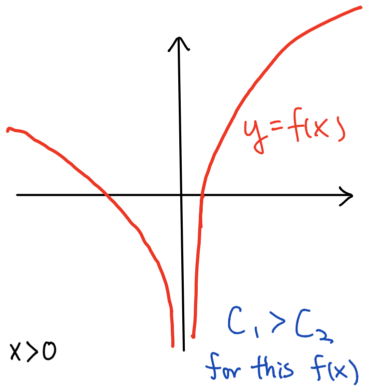
Q Find all $f(x)$ such that

$$f'(x) = \frac{1}{x}$$

on $\underbrace{\mathbb{R}\setminus\{0\}}_{\text{not an interval}}$.

Ans:

$$f(x) = \begin{cases} \ln x + C_1 & \text{for } x>0 \\ \ln(-x) + C_2 & \text{for } x<0 \end{cases}$$



Some basic integrals (a, b, k are constants)

$$\int (af(x) + bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^k\,dx = \frac{1}{k+1}x^{k+1} + C \quad \text{for } k \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\left.\begin{aligned} &\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C \\ &\int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C \end{aligned}\right\}$$

Ex Verify them by differentiation

Where is $\arccos x$?

Recall: $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

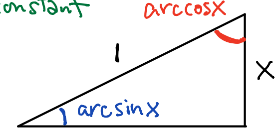
Q $\int \frac{-1}{\sqrt{1-x^2}} = \arccos x + C$?

or $-\arcsin x + C$?

A Both are correct!

$$\arccos x = -\arcsin x + \frac{\pi}{2}$$

differed by a constant



eg $\int \left(4x^2 + \csc^2 x - 3^x - \frac{2}{1+x^2}\right) dx$

can skip all steps

$$= 4\int x^2 dx + \int \csc^2 x \, dx - \int 3^x dx - 2\int \frac{dx}{1+x^2}$$

$$= 4\left(\tfrac{1}{3}x^3 + C_1\right) + \left(-\cot x + C_2\right) - \left(\frac{3^x}{\ln 3} + C_3\right) - 2\left(\arctan x + C_4\right)$$

$$= \frac{4}{3}x^3 - \cot x - \frac{3^x}{\ln 3} - 2\arctan x + C$$

where $C = 4C_1 + C_2 - C_3 - 2C_4$

Rmk

① We can simply introduce an integration constant C at the end instead of a C_i for each integral

② You may check your answer by differentiation

$$\frac{d}{dx}\left(\frac{4}{3}x^3 - \cot x - \frac{3^x}{\ln 3} - 2\arctan x + C\right)$$

$$= 4x^3 + \csc x - 3^x - \frac{2}{1+x^2}$$

$$= \text{integrand} \quad \checkmark$$

eg Suppose $f'(x) = x^{\frac{1}{3}} + 3$ on $\mathbb{R}$

$f(8) = 30$. Find $f(x)$.

Sol

$$f'(x) = x^{\frac{1}{3}} + 3$$

$$\Rightarrow f(x) = \int \left(x^{\frac{1}{3}} + 3\right) dx$$

$$= \frac{3}{4}x^{\frac{4}{3}} + 3x + C$$

Don't forget

Put $x = 8$,

$$f(8) = \frac{3}{4}(8)^{\frac{4}{3}} + 3(8) + C$$

$$30 = 12 + 24 + C$$

$$C = -6$$

$$\therefore f(x) = \frac{3}{4}x^{\frac{4}{3}} + 3x - 6$$

Integration by substitution

Let $f(u)$ be a function of u

$u = u(x)$ be a function of x

in terms of x in terms of u

$$\int f(u(x)) \frac{du}{dx}\, dx = \int f(u)\, du$$

Composition u = inner function   derivative of inner function

Rmk Proved from Chain rule:

If $g(u) = \int f(u)\, du$,

then $\frac{dg}{dx} = \frac{dg}{du} \cdot \frac{du}{dx}$

$= f(u) \frac{du}{dx}$

$= f(u(x)) \frac{du}{dx}$

eg $\int 2(3+2x)^5\, dx$

Sol Let $u = 3+2x$. $\frac{du}{dx} = 2$ $du = 2dx$

$\therefore \int 2(3+2x)^5\, dx = \int u^5\, du$

$= \frac{1}{6} u^6 + C$

$= \frac{1}{6}(3+2x)^6 + C$

eg $\int e^{2x^2+1} x\, dx$

Sol Let $u = 2x^2+1$ $\frac{du}{dx} = 4x$ $du = 4x\, dx$

$\therefore \int e^{2x^2+1} x\, dx = \frac{1}{4} \int e^u\, du$

$= \frac{1}{4} e^u + C$

$= \frac{1}{4} e^{2x^2+1} + C$

eg $\int x\sqrt{3x+1}\,dx$

Sol Let $u = 3x+1$ $\qquad \frac{du}{dx} = 3$

$x = \frac{1}{3}(u-1) \qquad du = 3dx$

$$\int x\sqrt{3x+1}\,dx$$

$$= \int \frac{1}{3}(u-1)\sqrt{u}\,\frac{1}{3}\,du$$

$$= \frac{1}{9}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$$

$$= \frac{1}{9}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C$$

$$= \frac{2}{45}(3x+1)^{\frac{5}{2}} - \frac{2}{27}(3x+1)^{\frac{3}{2}} + C$$

eg $\int \frac{\sin(1+\ln x)}{x}\,dx$

Sol Let $u = 1+\ln x \qquad du = \frac{1}{x}dx$

$$\therefore \int \frac{\sin(1+\ln x)}{x}\,dx = \int \sin u\,du$$

$$= -\cos u + C$$

$$= -\cos(1+\ln x) + C$$

Simpler presentation (without u)

$$\int \frac{\sin(1+\ln x)}{x}\,dx = \int \sin(1+\ln x)\,d(1+\ln x)$$

$$= -\cos(1+\ln x) + C$$

eg

$$\int \frac{dx}{7x-1} = \frac{1}{7}\int \frac{d(7x-1)}{7x-1} = \frac{1}{7}\ln|7x-1| + C$$

$\frac{d(7x-1)}{dx} = 7 \Rightarrow d(7x-1) = 7dx$

eg $\int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} dx$$

$$\left(\frac{d\cos x}{dx} = -\sin x \qquad d\cos x = -\sin x \, dx \right)$$

$$= \int \frac{-d\cos x}{\cos x}$$

$$= -\ln|\cos x| + C$$

or $= \ln|\cos x|^{-1} + C$

or $= \ln|\sec x| + C$

Rmk $\int \cot x \, dx = \ln|\sin x| + C$

eg $\int e^x \sec^2(e^x) \, dx$

$$= \int \sec^2(e^x) \, de^x$$

$$= \tan(e^x) + C$$

$$\boxed{\text{Substitution in Integration} \longleftrightarrow \text{Chain rule in Differentiation}}$$

Compare: For $x > 0$,

$$\frac{d}{dx}[\sin(\ln x)]^3$$

$$= 3\sin^2(\ln x)\cos(\ln x)\frac{1}{x}$$

(Differentiate layer by layer)

$$\int 3\sin^2(\ln x)\cos(\ln x)\frac{1}{x} dx$$

$$= \int 3\sin^2(\ln x)\cos(\ln x) \, d\ln x$$

$$= \int 3\sin^2(\ln x) \, d\sin(\ln x)$$

$$= \sin^3(\ln x) + C$$

(Integrate layer by layer)

Integration of Product of Trig. functions

Some useful formula :

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$$

$$\sin A \sin B = -\frac{1}{2}\left[\cos(A+B)-\cos(A-B)\right]$$

$$A=B \Rightarrow \begin{cases} \sin A\cos A = \frac{1}{2}\sin 2A \\ \cos^2 A = \frac{1}{2}(1+\cos 2A) \\ \sin^2 A = \frac{1}{2}(1-\cos 2A) \end{cases}$$

R.H.S. are easier to integrate

eg $\int \sin 5x \cos 3x \, dx$

$$= \int \frac{1}{2}(\sin 8x + \sin 2x)\,dx$$

$$= \frac{1}{2}\int \sin 8x\, dx + \frac{1}{2}\int \sin 2x\, dx$$

$$= \frac{1}{16}\int \sin 8x \, d(8x) + \frac{1}{4}\int \sin 2x \, d(2x)$$

$$= -\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + C$$

$$\frac{d(2x)}{dx} = 2$$

$$d(2x) = 2dx$$

$$dx = \frac{1}{2}d(2x)$$

Key: Lower the degree

eg

$$\int \cos x \cos^2 3x \, dx \qquad (\text{deg } 3)$$

$$= \int \cos x \cdot \frac{1}{2}(1+\cos 6x)\,dx \qquad (\text{deg } 2)$$

$$= \frac{1}{2}\int(\cos x + \cos x \cos 6x)\,dx$$

$$= \frac{1}{2}\sin x + \frac{1}{2}\int \frac{1}{2}(\cos 7x + \cos(-5x))\,dx$$

$$(\text{deg } 1)$$

$$= \frac{1}{2}\sin x + \frac{1}{28}\int \cos 7x \, d(7x)$$

$$- \frac{1}{20}\int \cos(-5x)\, d(-5x)$$

$$= \frac{1}{2}\sin x + \frac{1}{28}\sin 7x - \frac{1}{20}\sin(-5x) + C$$

$$= \frac{1}{2}\sin x + \frac{1}{28}\sin 7x + \frac{1}{20}\sin(5x) + C$$

$\boxed{\text{Find } \int \sin^m x \cos^n x \, dx}$

Express everything in terms of only $\cos x$ or $\sin x$

Case I : m is odd

Let $u = \cos x$. Then

$$\sin^2 x = 1 - \cos^2 x = 1 - u^2$$

$$\sin dx = -d\cos x = -du$$

eg

$$\int \sin^5 x \, dx$$

$$= \int \sin^4 x \sin x \, dx$$

$$= \int (1-u^2)^2 (-du)$$

$$= -\int (1 - 2u^2 + u^4) \, du$$

$$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= -\cos x + \frac{2}{3}\cos^2 x - \frac{1}{5}\cos^5 x + C$$

Case II : n is odd

Let $u = \sin x$. Then

$$\cos^2 x = 1 - \sin^2 x = 1 - u^2$$

$$\cos dx = d\sin x = du$$

eg

$$\int \sin^3 x \cos^3 x \, dx$$

$$= \int \sin^3 x \cos^2 x \cos x \, dx$$

$$= \int u^3 (1-u^2) \, du$$

$$= \int (u^3 - u^5) \, du$$

$$= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$$

$$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$

$m = 3$ is odd
so also case I
$\therefore u = \cos x$ also OK

or $\int \sin^3 x (1 - \sin^2 x) \, d\sin x$

$$= \int (\sin^3 x - \sin^5 x) \, d\sin x$$

$$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$

Case III: Both m, n are even

Apply formulas $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

to lower degree and reduce to Case I & II

eg $\int \sin^2 x\,dx$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{4}\int \cos 2x\,d(2x)$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Rmk Similarly

$$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

eg

$$\int \sin^2 x \cos^2 x\,dx$$

$$= \int \frac{1}{2}(1-\cos 2x)\cdot\frac{1}{2}(1+\cos 2x)\,dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x)\,dx$$

$$= \frac{1}{4}x - \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\,dx$$

$$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{32}\int \cos 4x\,d(4x)$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

A more difficult example:

eg $\int \sin^4 x \cos^2 x \, dx$

$$= \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 \left[\frac{1}{2}(1+\cos 2x)\right] dx$$

$$= \frac{1}{8}\int \left(1-\cos 2x-\cos^2 2x+\cos^3 2x\right) dx$$

$$= \frac{1}{8}x - \frac{1}{16}\sin 2x - \frac{1}{16}\int (1+\cos 4x)\,dx + \frac{1}{8}\int \cos^3 2x\,dx$$ (Case II)

$$= \frac{1}{8}x - \frac{1}{16}\sin 2x - \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{8}\left(\frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x\right) + C$$

$$= \frac{1}{16}x - \frac{1}{48}\sin^3 2x - \frac{1}{64}\sin 4x + C$$

$$\int \cos^3 2x \, dx$$

$$= \frac{1}{2}\int \cos^2 2x \cos 2x \, d(2x)$$

$$= \frac{1}{2}\int (1-\sin^2 2x)\, d\sin 2x$$

$$= \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right) + C$$

$$= \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + C'$$

Rmk Integrations using different trig identities result in very different looking answers